Problem Set 8

Macroeconomics III

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Problem 1

We have a Real Business Cycle (RBC) framework characterised by:

$$u(C_t, L_t) = \theta \log(C_t) + (1-\theta)\log(x_t) \tag{1}$$
$$Y_t = A_t K_t^{1-\alpha} L_t^{\alpha} \tag{2}$$
$$x_t + L_t = 1 \tag{3}$$
$$Y_t = C_t + I_t \tag{4}$$
$$I_t = K_{t+1} - (1-\delta)K_t \tag{5}$$

People discount future utility with β.

Isolate x_t in (3) and insert into (1) to get to utility depending on L_t:

$$u(C_t, L_t) = \theta \log(C_t) + (1-\theta)\log(1-L_t) \tag{6}$$

Isolate I_t in (4) and insert into (5). Insert the prod. function from (2):

$$\begin{aligned} K_{t+1} - (1-\delta)K_t = I_t &= Y_t - C_t \\ K_{t+1} + C_t &= Y_t + (1-\delta)K_t \\ K_{t+1} + C_t &= A_t K_t^{1-\alpha} L_t^{\alpha} + (1-\delta)K_t \qquad (7) \end{aligned}$$

Problem 1a - Lagrangian and FOCs

Set up the Lagrangian for this problem and find the FOCs

We set up the Lagrangian based on (6) and the constraint from (7):

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{l} \theta \log(C_t) + (1-\theta)\log(1-L_t) \\ +\lambda_t \Big[(1-\delta)K_t + A_t K_t^{1-\alpha} L_t^{\alpha} - K_{t+1} - C_t \Big] \end{array} \right) \tag{8}$$

We find the FOCs. The derivations follow the ones from the Ramsey model closely.

$$\begin{aligned}
C_t: &\qquad \frac{\theta}{C_t} = \lambda_t \\
L_t: &\qquad \frac{1-\theta}{1-L_t} = \lambda_t \alpha A_t K_t^{1-\alpha} L_t^{\alpha-1} \\
K_{t+1}: &\qquad \lambda_t = \beta \mathbb{E}_t \Big[\lambda_{t+1} \big((1-\alpha) A_{t+1} K_{t+1}^{-\alpha} L_{t+1}^{\alpha} + (1-\delta) \big) \Big] \\
\lambda_t: &\qquad K_{t+1} + C_t = (1-\delta)K_t + A_t K_t^{1-\alpha} L_t^{\alpha}
\end{aligned}$$

Problem 1a - Interpretation of FOCs (1/2)

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \qquad \Longrightarrow \qquad \begin{aligned} \beta^t \frac{\theta}{C_t} &= \beta^t \lambda_t \\ \frac{\theta}{C_t} &= \lambda_t \end{aligned}$$

The FOC wrt. C_t states that the marginal utility of consumption must be equal to the shadow price, λ_t, in the budget constraint. λ_t represents the utility gain in period t from a relaxation of the budget in period t.

$$\frac{\partial \mathcal{L}}{\partial L_t} = 0 \qquad \Longrightarrow \qquad \begin{aligned} \beta^t \frac{1-\theta}{1-L_t} &= \beta^t \lambda_t \alpha A_t K_t^{1-\alpha} L_t^{\alpha-1} \\ \underbrace{\frac{1-\theta}{1-L_t}}_{MU_x} &= \underbrace{\lambda_t}_{MU_c} \underbrace{\alpha A_t K_t^{1-\alpha} L_t^{\alpha-1}}_{\text{MPL}} \end{aligned}$$

The marginal utility of leisure should be equal to the product of marginal utility from consumption and the marginal product of labour. Intuitively, they gain the left hand side from increasing leisure, whereas they get can consume $w_t = MPL$ more if they work, which is multiplied with the marginal utility from that consumption.

Problem 1a - Interpretation of FOCs (2/2)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= 0 \\
\beta^t \lambda_t &= \mathbb{E}_t\left[\beta^{t+1}\lambda_{t+1}\left((1-\alpha)A_{t+1}K_{t+1}^{-\alpha}L_{t+1}^{\alpha} + (1-\delta)\right)\right] \\
\lambda_t &= \beta\mathbb{E}_t\left[\lambda_{t+1}\left((1-\alpha)A_{t+1}K_{t+1}^{-\alpha}L_{t+1}^{\alpha} + (1-\delta)\right)\right]
\end{aligned}
$$

This equation corresponds to the Euler equation. Inserting the marginal utility yields the Euler equation. The marginal utility from consuming today should be equal to the marginal utility of postponing consumption.

$$
\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \qquad \Longrightarrow \qquad K_{t+1} + C_t = (1-\delta)K_t + A_t K_t^{1-\alpha} L_t^{\alpha}
$$

This is simply the resource constraint of the economy. The right hand side is the resources available and the left hand side is the possible allocations of the resources. So how much do you have and how much do you spend.

Problem 1b - Set-up

We set capital depreciation such that

$$\delta = 1 \quad \Longrightarrow \quad K_{t+1} + C_t = A_t K_t^{1-\alpha} L_t^{\alpha}$$

We guess that the solution to this RBC model is given by:

$$\begin{aligned} L_t &= \bar{L} \\ K_{t+1} &= \rho_K A_t K_t^{1-\alpha} \\ C_t &= \rho_C A_t K_t^{1-\alpha} \end{aligned}$$

Where ρ_K and ρ_C are undetermined coefficients.

Method of undetermined coefficients:

1. Make a qualified guess about the form of the solution. The guesses include some unknown coefficients to give some flexibility.
2. Insert the solution into the FOCs.
3. If the guess is true, we can find an expression for each coefficient.
4. Insert the coefficients and verify that the FOCs are satisfied.

Problem 1b - To-do

$$\underbrace{\frac{\theta}{C_t}}_{\lambda_t} = \beta \mathbb{E}_t \left[\underbrace{\frac{\theta}{C_{t+1}}}_{\lambda_{t+1}} (1-\alpha) A_{t+1} K_{t+1}^{-\alpha} L_{t+1}^{\alpha} \right] \tag{9}$$

$$\frac{1-\theta}{1-L_t} = \lambda_t \alpha A_t K_t^{1-\alpha} \bar{L}^{\alpha-1} \tag{10}$$

$$K_{t+1} + C_t = A_t K_t^{1-\alpha} L_t^{\alpha} \tag{11}$$

Method of undetermined coefficients:

1. Make a qualified guess about the form of the solution. The guesses include some unknown coefficients to give some flexibility.
2. Insert the solution into the FOCs.
3. If the guess is true, we can find an expression for each coefficient.
4. Insert the coefficients and verify that the FOCs are satisfied.

$$L_t = \bar{L} \tag{12}$$

$$K_{t+1} = \rho_K A_t K_t^{1-\alpha} \tag{13}$$

$$C_t = \rho_C A_t K_t^{1-\alpha} \tag{14}$$

Problem 1b - Finding ρ_C and ρ_K (1/2)

Insert (13) and (14) into (11) and then use (12):

$$\begin{aligned}
A_t K_t^{1-\alpha} L_t^{\alpha} &= K_{t+1} + C_t \\
A_t K_t^{1-\alpha} L_t^{\alpha} &= \rho_K A_t K_t^{1-\alpha} + \rho_C A_t K_t^{1-\alpha} \\
L_t^{\alpha} &= \rho_K + \rho_C \\
\bar{L}^{\alpha} &= \rho_K + \rho_C \qquad (15)
\end{aligned}$$

Next, insert (14) into (9) while using (12):

$$\begin{aligned}
\frac{\theta}{C_t} &= \beta \mathbb{E}_t \left[\frac{\theta}{C_{t+1}} (1-\alpha) A_{t+1} K_{t+1}^{-\alpha} L_{t+1}^{\alpha} \right] \\
\frac{\theta}{\rho_C A_t K_t^{1-\alpha}} &= \beta \mathbb{E}_t \left[\frac{\theta}{\rho_C A_{t+1} K_{t+1}^{1-\alpha}} (1-\alpha) A_{t+1} K_{t+1}^{-\alpha} \bar{L}^{\alpha} \right] \\
\frac{\theta}{\rho_C A_t K_t^{1-\alpha}} &= \beta \mathbb{E}_t \left[\frac{\theta}{\rho_C K_{t+1}} (1-\alpha) \bar{L}^{\alpha} \right]
\end{aligned}$$

Problem 1b - Finding ρ_C and ρ_K (2/2)

Hence, we have the following:

$$\frac{\theta}{\rho_C A_t K_t^{1-\alpha}} = \beta \mathbb{E}_t \left[\frac{\theta}{\rho_C K_{t+1}} (1-\alpha) \bar{L}^{\alpha} \right]$$

We then insert (13) and use that $\mathbb{E}_t[A_t K_t] = A_t K_t$ as they are known:

$$\begin{aligned}
\frac{1}{\rho_C A_t K_t^{1-\alpha}} &= \beta \mathbb{E}_t \left[\frac{1}{\rho_C \rho_K A_t K_t^{1-\alpha}} (1-\alpha) \bar{L}^{\alpha} \right] \\
\frac{1}{\rho_C A_t K_t^{1-\alpha}} &= \beta \frac{1}{\rho_C \rho_K A_t K_t^{1-\alpha}} (1-\alpha) \bar{L}^{\alpha} \\
1 &= \beta \frac{1}{\rho_K} (1-\alpha) \bar{L}^{\alpha} \\
\rho_K &= \beta (1-\alpha) \bar{L}^{\alpha}
\end{aligned}$$

We can then use (15) to determine ρ_C:

$$\begin{aligned}
\rho_C &= \bar{L}^{\alpha} - \rho_K \\
\rho_C &= \big(1 - \beta(1-\alpha)\big) \bar{L}^{\alpha}
\end{aligned}$$

Problem 1b - Finding $\bar{L}$ (1/2)

Next, we proceed by looking at (10). We insert the guesses, (13) and (14), and use that $\lambda_t = \frac{\theta}{C_t}$:

$$\begin{aligned}
\frac{1-\theta}{1-\bar{L}} &= \lambda_t \alpha A_t K_t^{1-\alpha} \bar{L}^{\alpha-1} \\
&= \frac{\theta\alpha}{C_t} A_t K_t^{1-\alpha} \bar{L}^{\alpha-1} \\
&= \frac{\theta\alpha}{\rho_C A_t K_t^{1-\alpha}} A_t K_t^{1-\alpha} \bar{L}^{\alpha-1} \\
&= \frac{\theta\alpha \bar{L}^{\alpha-1}}{\rho_C} \\
&= \frac{\theta\alpha \bar{L}^{\alpha-1}}{\underbrace{\bar{L}^{\alpha}\left[1-\beta(1-\alpha)\right]}_{\rho_c}} \\
&= \frac{\theta\alpha}{\bar{L}\left[1-\beta(1-\alpha)\right]}
\end{aligned}$$

We now have an equation based on parameters and $\bar{L}$.

Problem 1b - Finding $\bar{L}$ (2/2)

We isolate $\bar{L}$ in the equation:

$$
\begin{aligned}
\frac{1-\bar{L}}{1-\theta} &= \frac{\bar{L}\big[1-\beta(1-\alpha)\big]}{\theta\alpha} \\
\frac{1-\bar{L}}{\bar{L}} &= \frac{(1-\theta)\big[1-\beta(1-\alpha)\big]}{\theta\alpha} \\
\frac{1}{\bar{L}} - 1 &= \frac{(1-\theta)\big[1-\beta(1-\alpha)\big]}{\theta\alpha} \\
\frac{1}{\bar{L}} &= 1 + \frac{(1-\theta)\big[1-\beta(1-\alpha)\big]}{\theta\alpha} \\
\frac{1}{\bar{L}} &= \frac{\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]}{\theta\alpha} \\
\bar{L} &= \frac{\theta\alpha}{\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]}
\end{aligned}
$$

Since, we know have $\bar{L}$, we can find the remaining variables.

Problem 1b - Overview

We have found the parameters, such that the solution is:

$$\bar{L} = \frac{\theta\alpha}{\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]}$$
$$\bar{L}^{\alpha} = \rho_C + \rho_K$$
$$\rho_C = \big(1-\beta(1-\alpha)\big)\bar{L}^{\alpha}$$
$$\rho_K = \beta(1-\alpha)\bar{L}^{\alpha}$$

1. The four equations must be able to all hold simultaneously - which they are.
2. It must be that $\bar{L} < 1$ such that $x_t = 1 - \bar{L} \geq 0$ - consumers don't have negative leisure (which would violate one of the conditions of the economy).

We find the equations characterizing consumption and capital:

$$C_t = \rho_C A_t K_t^{1-\alpha} = \big(1-\beta(1-\alpha)\big)\bar{L}^{\alpha} A_t K_t^{1-\alpha}$$
$$K_{t+1} = \rho_K A_t K_t^{1-\alpha} = \beta(1-\alpha)\bar{L}^{\alpha} A_t K_t^{1-\alpha}$$

Problem 1b - Steady state

In steady state we know that $A_t = 1$, $K_{t+1} = K_t = K$ and $C_t = C$. The steady state value of capital is given by:

$$
\begin{aligned}
K &= \beta(1-\alpha)\bar{L}^{\alpha} K^{1-\alpha} \\
K^{\alpha} &= \beta(1-\alpha)\bar{L}^{\alpha} \\
K &= \left[\beta(1-\alpha)\right]^{\frac{1}{\alpha}} \bar{L}
\end{aligned}
$$

As we know steady state K, we can find steady state C:

$$
\begin{aligned}
C &= \big(1-\beta(1-\alpha)\big)\bar{L}^{\alpha} K^{1-\alpha} \\
C &= \big(1-\beta(1-\alpha)\big)\bar{L}^{\alpha} \Big(\left[\beta(1-\alpha)\right]^{\frac{1}{\alpha}} \bar{L}\Big)^{1-\alpha} \\
C &= \big(1-\beta(1-\alpha)\big)\left[\beta(1-\alpha)\right]^{\frac{1-\alpha}{\alpha}} \bar{L}
\end{aligned}
$$

Problem 1c - Effect of $\beta \uparrow$ on $\bar{L}$

What is the effect of a marginal increase in the discount factor β on C and K? Provide an adequate interpretation.

We start by looking at $\bar{L}$, since it affects both K and C:

$$\bar{L} = \frac{\theta\alpha}{\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]}$$

Taking the derivative wrt. β yields:

$$\frac{\partial \bar{L}}{\partial \beta} = \frac{\alpha\theta(1-\theta)(1-\alpha)}{\Big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\Big)^2} > 0 \tag{16}$$

Hence, we see that $\bar{L}$ is increasing in β

Problem 1c - Effect of $\beta \uparrow$ on K

We then turn to K.

$$K = \left[\beta(1-\alpha)\right]^{\frac{1}{\alpha}} \bar{L} = \beta^{\frac{1}{\alpha}} (1-\alpha)^{\frac{1}{\alpha}} \bar{L}$$

We take the derivative with respect to β:

$$\frac{\partial K}{\partial \beta} = \frac{1}{\alpha} \beta^{\frac{1-\alpha}{\alpha}} (1-\alpha)^{\frac{1}{\alpha}} \bar{L} + \frac{\partial \bar{L}}{\partial \beta} \left[\beta(1-\alpha)\right]^{\frac{1}{\alpha}} > 0$$

K is increasing in β. An increase in β increases patience, which makes people save more and thus K increase. β also increases $\bar{L}$, which further increase K.

Problem 1c - Effect of $\beta \uparrow$ on C (1/2)

$$C = \big(1 - \beta(1-\alpha)\big)\big[\beta(1-\alpha)\big]^{\frac{1-\alpha}{\alpha}}\bar{L}$$

Taking the derivative of C is less straightforward. To ease dealing with the exponents we take the logarithm of C and take the derivative.[1]

$$Sign\Big(\frac{\partial \ln(f(x))}{\partial x}\Big) = Sign\Big(\frac{\partial f(x)}{\partial x}\Big) \tag{17}$$

We take the log of C:

$$\ln(C) = \ln\big(1 - \beta(1-\alpha)\big) + \frac{1-\alpha}{\alpha}\ln(\beta) + \frac{1-\alpha}{\alpha}\ln(1-\alpha) + \ln(\bar{L})$$

We then take the derivative wrt. β:

$$\frac{\partial \ln(C)}{\partial \beta} = -\frac{1-\alpha}{1-\beta(1-\alpha)} + \frac{1-\alpha}{\alpha\beta} + \frac{\partial \ln(\bar{L})}{\partial \beta}$$

We have already shown that $\frac{\partial \bar{L}}{\partial \beta} > 0$, why we know that $\frac{\partial \ln(\bar{L})}{\partial \beta} > 0$.

[1] We can do this because the natural logarithm is a strictly increasing function.

Problem 1c - Effect of $\beta \uparrow$ on C (2/2)

We look at the sign of the sums of the remaining terms:

$$-\frac{1-\alpha}{1-\beta(1-\alpha)} + \frac{1-\alpha}{\alpha\beta} = (1-\alpha)\Big(\frac{1}{\alpha\beta} - \frac{1}{1-\beta(1-\alpha)}\Big)$$

This is positive when:

$$\begin{aligned}\frac{1}{\alpha\beta} &> \frac{1}{1-\beta(1-\alpha)} \\ 1-\beta(1-\alpha) &> \alpha\beta \\ 1 &> \beta\end{aligned}$$

Which holds by assumption. Thus, we find that $\frac{\partial \ln(C)}{\partial \beta} > 0$ through two channels that are both positive.

Intuition: An increase in β decreases consumption initially but increases capital. The following increase in capital leads to higher steady state consumption. Furthermore, the increase in labour also leads to an increase in aggregate consumption.

Problem 1d - How does L depend on θ

$$\bar{L} = \frac{\theta\alpha}{\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]}$$

Taking derivative wrt. θ yields:

$$\begin{aligned}
\frac{\partial \bar{L}}{\partial \theta} &= \frac{\alpha\Big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\Big) - \alpha\theta\Big(\alpha - \big[1-\beta(1-\alpha)\big]\Big)}{\Big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\Big)^2} \\
&= \frac{\alpha(1-\theta)\big[1-\beta(1-\alpha)\big] + \alpha\theta\big[1-\beta(1-\alpha)\big]}{\Big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\Big)^2} \\
&= \frac{\alpha\big[1-\beta(1-\alpha)\big]}{\Big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\Big)^2} > 0
\end{aligned}$$

Intuition: When θ is large, the households value consumption more relative to leisure. θ enters the utility function and affects labour supply through that channel. Households will supply more labour if θ is high since supplying labour allows them to consume more.

Problem 1d - How does L depend on α

$$\bar{L} = \frac{\theta\alpha}{\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]}$$

Taking derivative wrt. θ yields:

$$\begin{aligned}
\frac{\partial \bar{L}}{\partial \alpha} &= \frac{\theta\big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big) - \alpha\theta\big(\theta + (1-\theta)\beta\big)}{\big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\big)^2} \\
&= \frac{\theta\big((1-\theta)\big[1-\beta(1-\alpha)\big) - \alpha\theta(1-\theta)\beta}{\big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\big)^2} \\
&= \frac{\theta(1-\theta)\big(1-\beta(1-\alpha) - \alpha\beta\big)}{\big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\big)^2} \\
&= \frac{\theta(1-\theta)\big(1-\beta\big)}{\big(\theta\alpha + (1-\theta)\big[1-\beta(1-\alpha)\big]\big)^2} > 0
\end{aligned}$$

Intuition: When α is large, labour is more productive and have a larger income share. Therefore, households are paid more for their labour, why they are willing to supply more labour. Hence, labour supply, $\bar{L}$, is increasing in α.